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1989 J. Phys. A: Math. Gen. 22 L671

(http://iopscience.iop.org/0305-4470/22/14/003)

J. Phys. A: Math. Gen. 22 (1989) L671–L672. Printed in the UK

LETTER TO THE EDITOR

On the calculation of the Fourier transform of a polygonal shape function

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Received 3 April 1989

Abstract. This letter gives a simple but creative closed-form solution of the Fourier transform of a polygonal shape function.

The aim of this letter is to provide a new closed-form solution for the two-dimensional Fourier transform of an uniform distribution over an N-sided polygon. The Fourier transform of shape functions is often encountered in optics [1] and electromagnetism [2].

Considering a polygon Σ in the xy plane as shown in figure 1, the shape function Σ is defined as

$$s(x, y) = \begin{cases} 1 & \text{if } (x, y) \text{ is in } \Sigma \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Then its two-dimensional Fourier transform is

$$s(u, v) = \iint_{-\infty}^{\infty} s(x, y) \exp[\mathrm{j}(ux + vy)] \, \mathrm{d}x \, \mathrm{d}y. \tag{2}$$

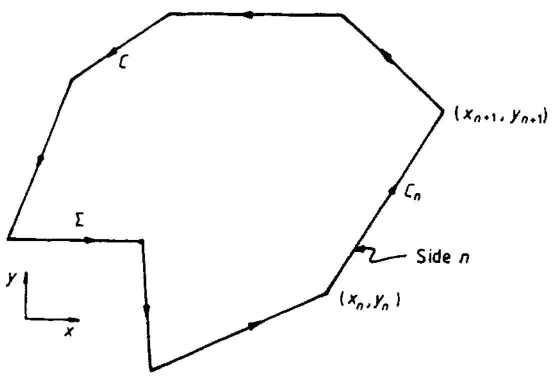

Figure 1. N-sided polygon. Line segment $C_n : x = a_n y + b_n$; refer to the text for parameters a_n, b_n.

The key problem is the determination of $S(u, v)$ from the coordinates (x_n, y_n) of each corner of the polygon Σ. Lee and Mittra [3] had given a closed-form solution of this issue. Unfortunately, the derivation of their result is very complex to follow. Alternatively, this letter gives a resourceful and succinct procedure, based on Stokes' theorem, to solve the same problem.

Define

$$A \triangleq \frac{1}{\mathrm{j}u} \exp[\mathrm{j}(ux + vy)]\boldsymbol{a}_y. \tag{3}$$

Then (2) can be written

$$S(u, v) = \iint_{\Sigma} \nabla \times \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} \tag{4}$$

where $\mathrm{d}\boldsymbol{S} = \mathrm{d}x\, \mathrm{d}y\, \boldsymbol{a}_z$.

Let $\mathrm{d}\boldsymbol{l} = \mathrm{d}x\, \boldsymbol{a}_x + \mathrm{d}y\, \boldsymbol{a}_y$ in figure 1; then, by Stokes' theorem

$$\iint_{\Sigma} \nabla \times \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} = \oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} \tag{5}$$

in which C is the boundary contour of Σ, and the line integral in (5) results in the closed-form solution:

$$\begin{aligned} S(u, v) &= \sum_{n=1}^{N} \int_{C_n} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} \\ &= \sum_{n=1}^{N} \frac{\exp(\mathrm{j}b_n u)\{\exp[\mathrm{j}(v + a_n u)y_{n+1}] - \exp[\mathrm{j}(v + a_n u)y_n]\}}{(\mathrm{j}u)\mathrm{j}(v + a_n u)} \end{aligned} \tag{6}$$

where

$$a_n = \frac{x_{n+1} - x_n}{y_{n+1} - y_n} \qquad \text{and} \qquad b_n = x_n - a_n y_n \tag{7}$$

are determined from the coordinates of the two corners of the line $x = a_n y + b_n$ of side n of the polygon in figure 1. The parameter a_n is the slope inversion of the line segment C_n. The closed-form solution appearing in (6) is much simpler than (6) in [3].

Although the Fourier transform solution presented is derived from a polygonal shape function, it can be applied to arbitrary shapes with continuous boundaries by approximating the curves by a finite number of straight-line segments.

References

[1] Kirkby P A and Thompson G H B 1976 *J. Appl. Phys.* **47** 4578
[2] Lee S W, Zarrillo G and Law C L 1982 *IEEE Trans. Antennas Propagat.* **AP-30** 904
[3] Lee S W and Mittra R 1983 *IEEE Trans. Antennas Propagat.* **AP-31** 99